THE MECHANICS OF VEHICLE COLLISIONS

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SUMMARY

Both in the treatment of safety problems and in the expert evaluation of motor vehicle accidents, the application of impact laws and the law of conservation of energy is an excellent, but much too little known tool. It is presented by means of some examples taken from practical situations and somewhat generalized.

1. Laws of Conservation of Momentum, Angular Momentum Energy

The law of conservation of momentum is derived from Newton's Law. Thus, for a point mass moving along a straight line, force equals mass times acceleration, $P = m \frac{dv}{dt}$

or $P\,dt = m\,dv$ which integrates to $\int P\,dt = m(v - v')$.

In words: The time integral of force is equal to the change in the motion variable (momentum).

Similarly, for angular motion with constant arm of the moment M is $J\,d\omega = M\,dt = P\,r\,dt$

or integrated, $r \int P\,dt = m\,i^2(\omega - \omega')$.

In words: The time integral of the torque is equal to the change in the moment of momentum (angular momentum).

The conservation of energy law also follows from Newton's basic law. We have $P\,ds = m \frac{dv}{dt}\,ds = m\,dv \cdot v = \frac{m}{2}\,d(v^2)$

or integrated, $\int P\,ds = \frac{m}{2}(v^2 - v'^2)$.

In words: The path integral of the force (work) is equal to the change in kinetic energy.

Angular motion of rigid bodies is associated with similar angular energy $J\omega^2/2$.. Since rectilinear and angular energies are scalar quantities, they can be added directly to give the total energy of the rigid body. On the other hand linear and angular momentum are vector quantities, which must be taken into account in addition.

2. Collision of Two Bodies

Two free bodies (i.e., without external forces) coming together can always be regarded as a system of point masses whose common center of gravity remains at constant velocity v_g as long as no external forces act on the system. This law holds before, during and after the collision since the forces which are set up in the process are internal forces for the system. Thus,

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_g \quad .$$

The behavior of the impacting bodies during and after the collision depends first on the type of impact (direct or oblique central, eccentric impact) and next on the elastic and plastic properties of the bodies at the impact point (completely or partially elastic, completely inelastic impact). In a completely inelastic central impact, the two bodies stick together and move in this way at their common speed until it is reduced by external resistance. Here the common speed u is attained with maximum compression at the impact point is identical to the speed of the entire body and to the speed v_g of the system's center of gravity. In a completely elastic impact, the bodies separate without having lost any of their total energy. In an eccentric impact, rotational motion will also arise.

3. Incompletely Elastic, Direct, Central Impact

Here $u = v_g$ and

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) u = m_1 v'_1 + m_2 v'_2 \quad .$$

The changes in momentum in the first phase of the impact up to maximum compression are

$$m_1 (v_1 - u) = - m_2 (v_2 - u) = S_1 \quad .$$

In the second phase, rebound, the changes in momentum are

$$m_2 (v'_2 - u) = - m_1 (v'_1 - u) = S_2 \quad .$$

If we call the ratio of the momentum changes the impact coefficient, then

$$k = S_1/S_2 = \frac{u - v'_1}{v_1 - u} = \frac{v'_2 - u}{u - v_2} \; ;$$

from the equation for conservation of momentum we get

$$u = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{m_1 v'_1 + m_2 v'_2}{m_1 + m_2} \quad .$$

Substitution yields

$$k = \frac{v'_2 - v'_1}{v_1 - v_2} \quad .$$

If, after the collision, the bodies move together (completely inelastic impact), then $v'_2 - v'_1 = 0$ or $k = 0$; if the bodies are completely elastic, then $k = 1$. Actually bodies are "incompletely elastic," i.e., at the instant of maximum compression part of their total energy has already been used up in distortions (permanent deformations), and part is stored elastically in the compression region and is transformed into kinetic energy during rebound. Thus the bodies have a different final velocity, and they separate after the impact. The final velocities are

$$v'_1 = u - \frac{k\,(v_1 - v_2)\,m_2}{m_1 + m_2}\,; \quad v'_2 = u + \frac{k\,(v_1 - v_2)\,m_1}{m_1 + m_2}\,.$$

$$v'_2 - v'_1 = k\,(v_1 - v_2)$$

4. <u>Change in Energy</u>

The decrease in kinetic energy of the two bodies after direct central impact is

$$E_v = m_1 \frac{v_1^2 - v'^2_1}{2} + m_2 \frac{v_2^2 - v'^2_2}{2} = (1 - k^2)\,\frac{m_1 m_2}{m_1 + m_2}\,\frac{(v_1 - v_2)^2}{2}\,.$$

It is especially noteworthy that the energy difference of each of the two bodies before and after the collision depends on the difference in the squares of the velocities $(v^2 - v'^2)$, but the total loss of kinetic energy depends on the square of the absolute velocity difference of the two bodies before the collision $(v_1 - v_2)^2$. For a completely elastic impact with $k = 1$, the energy loss is zero.

In a completely inelastic impact of two equal masses, the kinetic energy loss with $k = 0$ is

$$E_v = \frac{m}{4}\,(v_1 - v_2)^2;$$

the original total kinetic energy

$\frac{m}{2}\,(v_1^2 + v_2^2)$ has been reduced to $\frac{m}{4}\,(v_1 + v_2)^2$.

If the second body is at rest before the impact, then $v_2 = 0$

and $E_v = \frac{1 - k^2}{2}\,v_1^2\,\frac{m_1 m_2}{m_1 + m_2}$.

If the second body is an immovable wall, then $m_2 = \infty$, and $v_2 = 0$; hence $E_v = \frac{mv^2}{2}\,(1 - k^2)$. In the case of complete elasticity, the moving body will rebound from the wall at full speed; if it is completely inelastic, then it will remain at the wall and its total energy will be transformed into deformation energy. The equation says nothing, however, about how the deformation is distributed between the moving bodies and the wall.

5. Impact Coefficient for Head-on Impact; Vehicle Spring Diagram

Only few numerical data are known concerning the actual elastic and plastic behavior of motor vehicles. In order to derive a representation, the deceleration diagram as given in (1) (pg. 305) for a heavy American passenger car during impact against a solid wall was analyzed. The measured impact decelerations b are given as a function of time t, Figure 1. Integration gives $\int b\, dt = 17{,}0$ m/s . The measured impact velocity was 48.3 km/hr = 13.42 m/sec. Assuming partially elastic impact, the value of the integral is 17.0 = $(1 + k)v_0 = (1 + k)\ 13{,}42$ or $k = 0.265$.

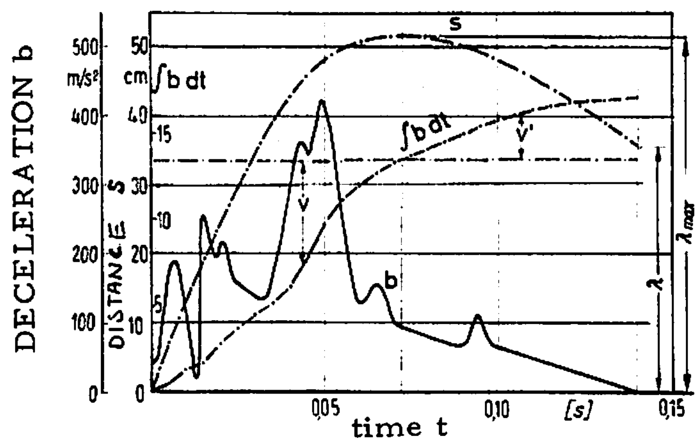

Figure 1 Time History of Impact of a Heavy Passenger Car Against a Solid Wall; After (1)

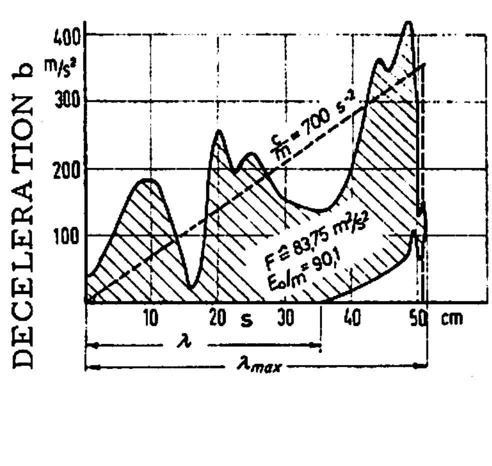

Figure 2 Spring Diagram of the Car of Figure 1

Recoil of the car begins after an impact interval of about 0.07 sec. Repeated integration of the velocity $v_0 - \int b\,dt$ derived in this way yields the deformation distance s. At the point of reversal, its peak value is 51 cm while at the end of the impact the permanent deformation distance was 35.5 cm. This agrees approximately with the data of Figure 13 of the cited paper (1) where the permanent deformation for this speed is given as about 40 cm.

If we now plot acceleration vs. deformation distance s, we get the "spring diagram" of the car which corresponds physically to the force-compression curve of an energy dissipating spring. Its area thus represents the work lost per unit mass. The initial kinetic energy of the impacting car was $v_0^2/2 = 90{,}1\ m^2/s^2$ per unit mass while the diagram area was $83{,}75\ m^2/s^2$, which again corresponds to $k = 0.265$.

The strong force fluctuations during compression correspond to the collapse, one after another, of different parts of the car; since, according to (1), these force fluctuations at the impact point are not propagated to the center of gravity, we are obviously justified in assuming a smoothed force rise for the deceleration of the center of gravity. Such a smoothed spring diagram with the same area is shown dotted in Figure 2. The spring constant per unit mass is $c/m = 700\ sec^{-2}$. This value is also the square of the circular frequency for the first impact phase. It gives a compression time of 0.06 sec, somewhat shorter than the value determined in Figure 1.

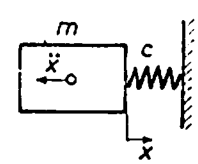

Figure 3 Schematic for Impact Against a Solid Wall

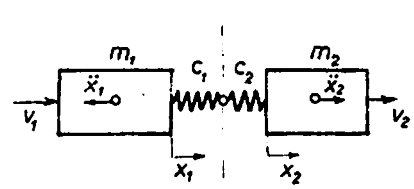

Figure 4 Schematic for Direct Collision of Two Cars Traveling in the Same Direction

If this analysis is not absolutely exact and if the resulting impact coefficient appears somewhat too high, nevertheless, consideration shows that, for establishing physically reliable data -- e.g., for expert accident evaluation -- one can begin with a linear spring diagram and the collision can be treated as an oscillation problem.

6. Oscillation Calculation for Head-on Impact

A schematic of the oscillating system is shown in Figure 3; in any case energy dissipating properties must be ascribed to the spring. The process falls into two phases, namely the impact from the instant of contact to maximum compression, and recoil until contact stops. For both phases we have the same oscillation equation but with different spring constants c:

$$\ddot{x} + \frac{c}{m}x = 0 \quad \text{with} \quad \frac{c}{m} = \omega^2 \quad \text{and} \quad P = cx$$

and with solution

$$x = a\sin\omega t \quad \text{(deformation distance)}$$
$$\dot{x} = a\omega\cos\omega t \quad \text{(velocity)}$$
$$\ddot{x} = -a\omega^2\sin\omega t \quad \text{(deceleration)}.$$

For the first phase of the above example the force increase is given approximately by $c/m = \omega^2 = 700\ s^{-2}$; at the instant of contact the time is $t = 0$ and the initial velocity is $\dot{x} = v_0 = 13{,}42\ m/s = a\omega$ hence $a = 0.508$; this is the maximum deformation which is attained at $t = t'$ together with the maximum deceleration $\ddot{x}_{max} = -a\omega^2 = -0{,}508 \cdot 700 = 355{,}6\ m/s^2$.

The weight of a heavy passenger car is taken to be 1500 to 2000 kg; thus the maximum collision force is about 53 to 71 tons. A passenger who is not strapped in does not feel this impact peak; instead, he flies forward at almost initial velocity of 13.42 m/sec and strikes against some part of the car which has stopped abruptly. In this process, his head alone, with a weight of about 4 kg, has a kinetic energy of about 32 mkg. 7mkg, however, can be enough for a skull fracture.

According to the assumed idealized spring diagram, the impact force drops abruptly at the instant of maximum compression. For calculating the recoil phase we use the smaller spring coefficient c' and the corresponding spring force $P' = c'(x - \lambda)$. In the spring diagram, the area under the spring expansion line represents the potential energy which is transformed back into kinetic energy.

7. Oscillation Calculation for Head-on Collision

The collision of two cars traveling in the same or in opposite directions can be treated in a similar way as an oscillation problem. The simplest example is the collision of two equal cars traveling in the same direction with one behind the other; this is considered according to the schematic diagram of Figure 4. The further simplifying assumption is made that the elastic and plastic characteristics of the colliding part of both cars are identical, $c_1 = c_2$. In the more general case, a reduced spring coefficient $C = \frac{c_1 c_2}{c_1 + c_2}$ would be introduced. If other force effects (e. g. due to braking or propulsion) can be neglected because of the shortness of the impact and the magnitude of the impact forces, then for equal car masses their decelerations will always be equal, $\ddot{x}_1 = -\ddot{x}_2$. The equations of motion are then $m\ddot{x}_1 = -m\ddot{x}_2 = c\frac{x_2 - x_1}{2}$.

From this we get $x_1^{IV} + \frac{c}{m}\ddot{x}_1 = 0$ with $\frac{c}{m} = \omega^2$.

With $m_1 = m_2 = m$ and $c_1 = c_2 = c$, this oscillation equation has the following solutions:

First phase:

$$\ddot{x}_1 = -a \sin\omega t \qquad \ddot{x}_2 = +a \sin\omega t$$

$$\dot{x}_1 = v_{01} - \frac{a}{\omega}(1 - \cos\omega t) \qquad \dot{x}_2 = v_{02} + \frac{a}{\omega}(1 - \cos\omega t)$$

$$x_1 = (v_{01} - \frac{a}{\omega})t + \frac{a}{\omega^2}\sin\omega t; \qquad x_2 = (v_{02} + \frac{a}{\omega})t - \frac{a}{\omega^2}\sin\omega t .$$

Further, assume that both car masses are $m = 200$ kgs²/m , the initial speeds are $v_{01} = 40$ km/h $= 11{,}11$ m/s and $v_{02} = 20$ km/h $= 5{,}55$ m/s and the spring coefficient corresponding to the above example is $c = 140,000$ kg/m. We then get for the first phase

$$\frac{c}{m} = \omega^2 = 700;\ \omega = 26{,}46;\ t' = \frac{\pi}{2\omega} = 0{,}0594$$

$$\ddot{x}_1 = -73{,}45 \sin\omega t \qquad \ddot{x}_2 = +73{,}45 \sin\omega t$$

$$\dot{x}_1 = 2{,}78 \cos\omega t + 8{,}33 \qquad \dot{x}_2 = -2{,}78 \cos\omega t + 8{,}33$$

$$x_1 = 0{,}105 \sin\omega t + 8{,}33\, t; \qquad x_2 = -0{,}105 \sin\omega t + 8{,}33\, t\ .$$

At the end of the phase the compression at $t' = 0{,}0594$ seconds is $\sin\omega t = 1$ and $\cos\omega t = 0$; hence $P_{max} = 200 \cdot 73{,}45 = 14\,690$ kg, $\dot{x}_1 = \dot{x}_2 = 8{,}33$ m/s, $x_2 = 8{,}33 \cdot 0{,}0594 - 0{,}105 = 0{,}389$ m, $x_1 = +8{,}33 \cdot 0{,}0594 + 0{,}105 = 0{,}599$ m and $(x_1 - x_2) = 0{,}210$ m.

A check again gives $P_{max} = c\,(x_1 - x_2)/2 = 14\,690$ kg. The area under the characteristic line of both springs $P_{max}\,(x_1 - x_2)/2 = 14\,690 \cdot 0{,}105 = 1542{,}5$ mkg corresponds to the value $\frac{m}{4}\,(v_1 - v_2)^2$. If we assume an impact coefficient of $k = 0.2$, then the total lost energy due to deformation is $E_v = m\,(1 - k^2)\,(v_1 - v_2)^2/4 = 200 \cdot 0{,}96 \cdot 5{,}55^2/4 = 1478{,}4$ mkg.

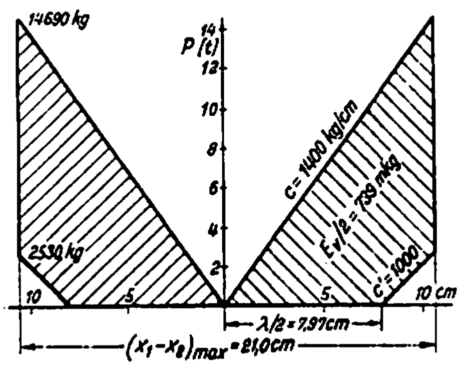

Figure 5 Spring Diagram for the Collision of Two Equal Cars With Equal Elastic Characteristics

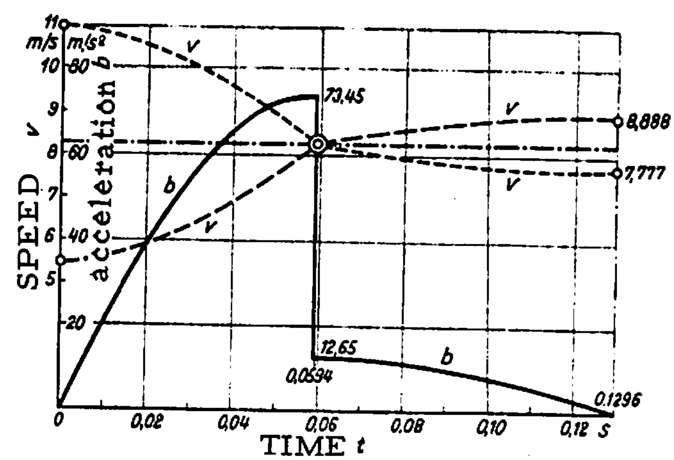

Figure 6 Time History of the Collision of Figure 5

The area under the spring expansion line of the spring diagram of a car must correspond to half the difference, i.e., $1/2\,(1542{,}5 - 1478{,}4) = 32{,}5$ mkg. If the spring expansion constant is taken as 100,000 kg/m, than the permanent deformation will be 0.105 - 0.0253 = 0.0797 m and the maximum spring expansion force will be 2530 kg. Thus the spring diagram can be confirmed. With the above assumptions, we have $v'_2 - v'_1 = k\,(v_1 - v_2) = 0{,}2 \cdot 5{,}55 = 1{,}11$ m/s.

With equal masses this difference coincides uniformly with the common speed of $u = 8.333$ m/sec., i.e. $v'_2 = 8.888$ m/sec and $v'_1 = 7.777$ m/sec. For many purposes, this result is already satisfactory.

Integration of the second phase and determination of the time history start with the equation

$$m\ddot{x}_1 = -m\ddot{x}_2 = c'(x_1 - x_2 - \lambda)/2$$

which leads to

$$\begin{aligned} \ddot{x}'_1 &= -12{,}65\cos\Omega t & \ddot{x}'_2 &= +12{,}65\cos\Omega t \\ \dot{x}'_1 &= -0{,}566\sin\Omega t + 8{,}333 & \dot{x}'_2 &= +0{,}566\sin\Omega t + 8{,}333 \\ x'_1 &= +0{,}0253\cos\Omega t + & x'_2 &= -0{,}0253\cos\Omega t + \\ &\quad + 8{,}333t + K_1; & &\quad + 8{,}333t + K_2 \,. \end{aligned}$$

The time measurement for the second phase is started over from zero; it takes t'' seconds. At $t = t''$ we have $\cos\Omega t = 0.$ and $\sin\Omega t = 1$, and also $\Omega t'' = \pi/2;$ with $\Omega = \sqrt{c'/m} = \sqrt{500} = 22{,}35$. We thus have $t'' = 0.0702$ seconds.

The constants K_1 and K_2 are determined from the final values of the first phase $x'_{10} = 0.599$ and $x'_{20} = 0.389$ for $\cos\Omega t' = 1$; they are $K_1 = 0.5737$ and $K_2 = 0.4143$. Thus at $t = t''$, the distance is

$$\begin{aligned} x''_1 &= 8{,}333 \cdot 0{,}0702 + 0{,}5737 = 1{,}1570 \text{ m} \\ x''_2 &= 8{,}333 \cdot 0{,}0702 + 0{,}4143 = 0{,}9976 \text{ m} \end{aligned}$$

and, as assumed, $x''_1 - x''_2 = 0.1594$ m is the total deformation of both cars. The collision process is shown in Figures 5 and 6.

8. Chain Reaction Collision

It has already been indicated that the degree of destruction in the direct central impact of two bodies is proportional to the square of the (absolute) velocity difference before the impact. Thus we can always start with the relative velocity and consider the impact as if one of the bodies was at rest before the impact. This approach is also useful in clarifying the processor in chain reaction collisions which have recently become common on autobahns. Here it is often difficult to determine which of a number of collisions which take place in a column in a very short time period is the first and original one. As a rule these column collisions take place in poor visibility when the following driver notices too late that the car in front has rapidly reduced speed for some reason.

As an example for estimating the resulting destruction, we shall first use the limiting case of the simultaneous collision of three cars (of equal size) whose velocity difference is, say, 20 km/hr. The speeds could thus be, for example, 50...30...10...km/hr, Figure 7. In this case the middle vehicle can be thought of as at rest, and the two others move toward it with equal and opposite velocities of 20 km/hr. If all three vehicles have the same mass m and the same elastic-plastic characteristics, then the middle vehicle acts like a (partially elastic) wall; its velocity does not change as a result of the impact.

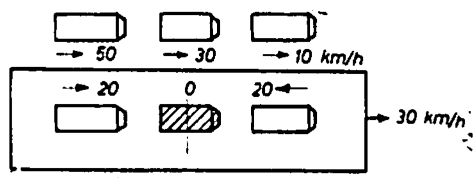

Figure 7 Absolute and Relative Velocity of Three Cars for Explanation of Chain Reaction Collision

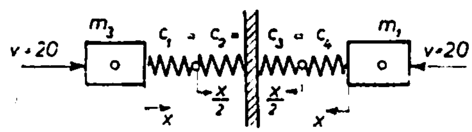

Figure 8 Schematic of Simultaneous Collision of Three Cars for Figure 7

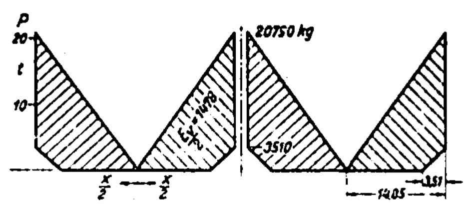

Figure 9 Spring Diagram of Simultaneous Collision of Three Cars for Figures 7 and 8

For each of the two impacts, the equivalent oscillation diagram of Figure 8 holds with a spring rate of $C = c/2d$ and the oscillation equation

$$\ddot{x} + \frac{C}{m} x = 0.$$

The frequency is

$$\omega = \sqrt{\frac{C}{m}} = \sqrt{\frac{70000}{200}} = \sqrt{350} = 18{,}7.$$

For the compression phase of the process we have the equations

$$x = a \sin \omega t; \quad \dot{x} = a\omega \cos \omega t; \quad \ddot{x} = -a\omega^2 \sin \omega t$$

at $t = 0$, $x = 0$ and $\dot{x} = v_0 = 5{,}55$ m/s;

whence $a = 5{,}55/18{,}7 = 0{,}297$ m.

At maximum compression $t = t'$; $x_{max} = a$, $\dot{x} = 0$ and $-\ddot{x}_{max} = +a\omega^2 = 0{,}297 \cdot 350 = 103{,}95$ m/s^2 The resulting maximum force is then $P_{max} = m\ddot{x}_{max} = Cx_{max} = 20790$ kg we again have the relation for the transformed initial energy $E_0 = mv_0^2/2 = P_{max}x_{max}/2 = 3080$ mkg . With an impact coefficient of $k = 0.2$, the deformation energy which is converted into kinetic energy after recoil is $E_1 = 0{,}2^2 \cdot E_0 = 123{,}2$ mkg .

With the recoil spring coefficient of the previous example $c' = 100{,}000$ kg/m, the permanent deformation is 11.34 mm. The spring diagram is shown schematically in Figure 9. As can be seen, displacements, accelerations and forces are larger than in the previous example because the middle car cannot move with the impact. The energy absorbed in deformation in this double collision is twice 2957 mkg, i.e., each of the two impacts gives twice as much destruction as a single impact with the same velocity difference. This would also be the case if the first and third cars collided without the second car being present, but with a different distribution. It has been assumed here that the impact coefficient is constant independent of the amount of compression, which is probably not the case.*

This comparison of the separated with the simultaneous collisions for velocity differences of 20 km/hr is no longer permissible if the collisions of the cars are separate even though they follow one another very closely. In that case they no longer take place with velocity differences of 20 km/hr. If the impacts are almost completely inelastic and if, for example, car 3 strikes car 2 first, then both end up with a speed of about 40 km /hr and hence, car 2 strikes car 1 with a velocity difference of 30 km/hr. On the other hand, if car 2 strikes car 1 first, then both end up with a speed of about

*In this connection, see the paper by Prof. Eberan-Eberhorst, ATZ 1961, p. 272, Figure 15 which appeared after completion of this manuscript.

20 km/hr and car 3 hits them at 50 km/hr. If we ignore the possibility of repeated collisions, then the sum of the destruction energies is

in the first case

$$E_v = \frac{m}{4}(1-k^2)\ (13{,}88-8{,}33)^2 + (11{,}11-2.77)^2$$

and in the second case

$$E_v = \frac{m}{4}(1-k^2)\ (13{,}88-5{,}55)^2 + (8{,}33-2{,}77)^2$$

or 4200 mkg in both cases compared with 2957 mkg for two separate collisions with velocity differences of 20 km/hr.

The time history of such multiple collisions can also be investigated using the above methods if the initial velocities are known precisely enough. Unfortunately, this is not generally the case. Also the tackographs installed in trucks and buses generally yield no usable velocity records after the first impact. Further, these considerations show clearly that with multiple collisions the visible destruction can be far greater than experience with single collisions would suggest, or conversely: there is the danger of over-estimating the speeds of chain reaction collisions.

9. Right-Angle Collision of Motor Vehicles

If, for example, two motor cars collide at right angles at a crossroad and if the front of vehicle 1 strikes the side of vehicle 2, then at the impact point there is a resultant impact force of ± P whose direction at the instant of contact is given by the direction of the relative velocity v_{rel}. For the special case shown in Figure 10a, the impact force + P acting on car 2 passes through its c. g.; hence for vehicle 2 we are dealing with an oblique central impact. The reaction force - P acting on vehicle 1, however, misses its c.g.; thus it experiences an oblique eccentric impact. Consequently, the subsequent motion is different for the two vehicles; vehicle 2 is first shoved along, without rotation, in the direction of the common velocity u at the impact point while vehicle 1 experiences an additional clockwise rotation which absorbs some energy.

In order to calculate the magnitude and direction of the velocities after impact, the assumption is made that the friction forces on the wheels are vanishingly small relative to the impact forces, hence that external forces need not be considered while the impact takes place. It is further assumed that the impact interval is infinitely short so that the vehicles carry out no significant motion during this time. This assumption must later be checked with the result.

The line of action of the force + P can be shifted to the c.g. of vehicle 2 and broken into components P_1 and P_2 in the original directions of motion P_1 causes a displacement normal to the original direction of motion while P_2 decreases the original forward velocity v_2. There is no rotation of vehicle 2. The force - P is decomposed at the impact point into the corresponding opposite components. - P then passes through the c.g. of vehicle 1 and decreases its original speed v_1. If we also apply the forces $\pm P_2$, at the c.g. S_1, then it is seen that P_2 causes a displacement normal to the original direction of motion of vehicle 1, and also that the force couple pP_2 results in rotation about the c.g. S_1.

This force decomposition is valid for each instant of the impact, and hence also for the time integral of the impact components. Thus the law of conservation of momentum can be applied. Hence, for vehicle 2

$$\int P_1 dt = M_2 v'_{21}; \int P_2 dt = M_2 (v_2 - v'_{22}).$$

For vehicle 1 we have

$$\int P_1 dt = M_1 (v_1 - v'_{11}); \int P_2 dt = M_1 v'_{12}.$$

Further, for the force couple on vehicle 1

$$p \int P_2 dt = M_1 i^2{}_2 \omega_1 \quad \text{or} \quad \int P_2 dt = M_1 \frac{i^2{}_1}{p} \omega_1 .$$

Equating appropriate terms gives

$$\int P_1 dt = M_2 v'_{21} = M_1 (v_1 - v'_{11})$$
$$\int P_2 dt = M_1 v'_{12} = M_2 (v_2 - v'_{22}) = M_1 \frac{i^2{}_1}{p} \omega_1.$$

(This relation is given incorrectly in (2).)

Further, under the above assumptions, the velocity v_g of the system c.g. S_g remains constant, and we have the vector equation

$$M_1 v_1 \hat{+} M_2 v_2 = (M_1 + M_2) v_g = M_1 v'_1 \hat{+} M_2 v'_2 .$$

Finally, under the assumption of "complete impact" the impact points of both vehicles at the instant of greatest compression must have the same velocity u. For central impact on vehicle 2, this is identical with the translational velocity v'_2 of the c.g. S_2 and of the entire vehicle 2. For vehicle 1 we have the vector equation (at the impact point)

$$u = v'_{11} \hat{+} v'_{12} \hat{+} \omega_1 p \;.$$

Using these relationships and with the simplifying assumption that $M_1 = M_2 = m$, Figure 10b shows the vector diagram of the momentum and velocities for $v_1 = 12$ and $v_2 = 8$ m/s, $m = 100$ kgs^2/m. The relations now simplify to

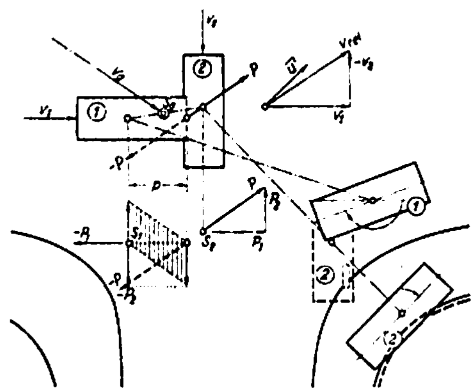

Figure 10a Right Angle Collision at Crossroad; Location of Collision and Final Positions of the Cars; Decomposition of Impact Force

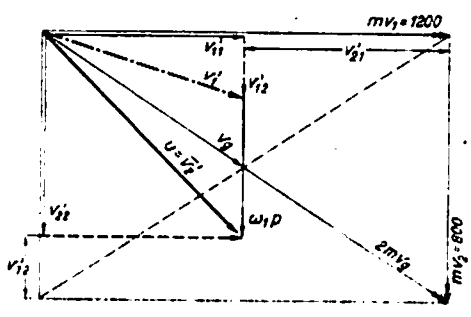

Figure 10b Vector Diagram of Velocities and Moments for Figure 10a

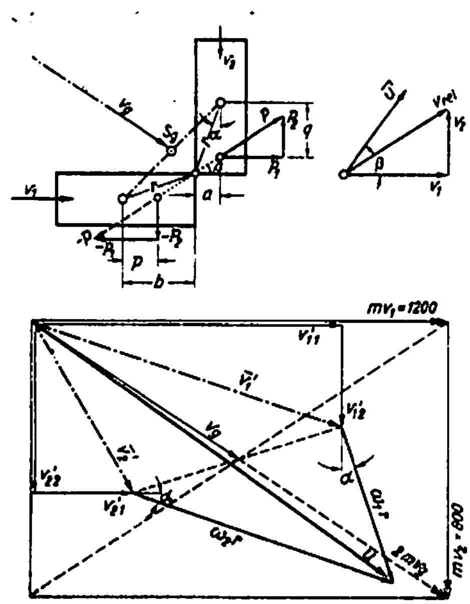

Figure 11 Limiting Case of Right-Angle Collision; Above: Impact Force Decomposition; Below: Vector Diagram of Velocities and Momenta

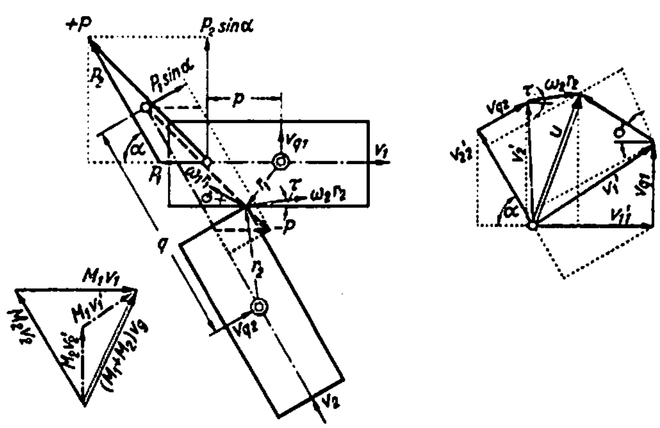

Figure 12 Oblique Collision; Left: Vector Diagram of Momenta; Center: Decomposition of Impact Force; Right: Vector Diagram of Velocities

For numerical calculation it was assumed that i = 1.3 and p = 2.0.

No assumption as to the impact direction was directly required in order to calculate the velocities; a requirement was that the force, as a vector, was in the direction of the c.g. of vehicle 2. Actually, the assumption of infinitely short impact time and hence the constancy of the direction of P are not strictly correct. At the instant of contact P is in the direction of v_{rel}; at the instant of maximum compression when the impact points of both cars have velocity u and hence no tangential forces can be transmitted, the direction of P must obviously be perpendicular to the direction of u.

The further motion of the vehicles until they come to rest is discussed below.

Another limiting case of right-angle collision is shown in Figure 11. For equal masses of the two vehicles we have

$$mv_1 \hat{+} mv_2 = 2mv_g; \quad v_g = \frac{v_1 \hat{+} v_2}{2} = \frac{v'_1 \hat{+} v'_2}{2}$$

momentum laws yield

$$\int P_1 dt = mv'_{21} = m(v_1 - v'_{11}) = \frac{J}{q}\omega_2$$

$$\int P_2 dt = mv'_{12} = m(v_2 - v'_{22}) = \frac{J}{p}\omega_1 \quad .$$

In determining the distances $p = b - a/tg\beta$ and $q = b - a \cdot tg\beta$ the impact direction must be assumed. If we choose for this the direction of v_{rel}, then, from what has been said above, there is an error whose magnitude must be checked later against the result and which may have to be improved in an iterated calculation.

Under the assumption that, at the instant of maximum compression, the rotational velocities ωr of the two cars at the impact points are perpendicular to the directions r, we get the following additional relationships from the velocity diagram

$$\begin{aligned} v'_{11} + \omega_1 r \sin\alpha &= v'_{21} + \omega_2 r \cos\alpha \\ v'_{22} + \omega_2 r \sin\alpha &= v'_{12} + \omega_1 r \cos\alpha \end{aligned} \qquad \text{with} \qquad tg\alpha = a/b \quad .$$

Thus, all velocity components can be calculated; the angular velocities are derived from

$$\omega_1\left[r\sin\alpha-\frac{r\cdot\cos\alpha+2i^2/p}{r\cdot\sin\alpha}\left(\frac{2i^2}{q}+r\cdot\cos\alpha\right)\right]=-v_1-\frac{v_2}{r\cdot\sin\alpha}\left(\frac{2i^2}{q}+r\cdot\cos\alpha\right)$$

$$\omega_2\left(r\cos\alpha+\frac{2i^2}{q}\right)=v_1+\omega_1\cdot r\cdot\sin\alpha\ .$$

The difference in direction between v_{rel} and the velocity rotated 90° $\overline{u'}$ again shows the uncertainty in the assumption of an average momentum direction which , in any given case, can be corrected by repeated calculation.

10. Oblique Collision

The solution of this problem using the same method is presented in the example of Figure 12. After the impact forces P are shifted to the longitudinal axis of the car and the components are determined along the two original directions of motion, we get for vehicle 1

$$\int P_1dt+\cos\alpha\int P_2dt=M_1(v_1-v'_{11})$$
$$p\cdot\sin\alpha\int P_2dt=J_1\omega_1=M_1i^2_1\omega_1;\ \sin\alpha\int P_2dt=M_1v_{1q}\ .$$

Vehicle 2

$$\int P_2dt+\cos\alpha\int P_1dt=M_2(v_2-v'_{22})$$
$$q\cdot\sin\alpha\int P_1dt=J_2\omega_2=M_2i^2_2\omega_2;\ \sin\alpha\int P_1dt=M_2v_{2q}\ .$$

The velocities v_q are always perpendicular to the original direction of motion.

From the velocity diagram we get for the components in direction 1

$$v'_{11}+v'_{22}\cos\alpha=v_{2q}\sin\alpha+\omega_1r_1\cos\sigma+\omega_2r_2\cos\tau$$

in direction 2

$$v'_{22}+v'_{11}\cos\alpha=v_{1q}\sin\alpha+\omega_1r_1\cos(\alpha-\sigma)+\omega_2r_2\cos(\alpha+\tau)\ .$$

From these relations the velocity components can be calculated. They are

$$\omega_1=\frac{p}{i^2_1}v_{1q}\qquad\omega_2=\frac{q}{i^2_2}v_{2q}$$

$$v_{1q}=\frac{M_2}{M_1\sin\alpha}(v_2-v'_{22})-\frac{1}{\mathrm{tg}\,\alpha}(v_1-v'_{11})$$

$$v_{2q} = \frac{M_1}{M_2 \sin\alpha}(v_1 - v'_{11}) - \frac{1}{\operatorname{tg}\alpha}(v_2 - v'_{22}) \,.$$

With the abbreviations

$$a = \frac{p\,r_1}{i^2_1}\,\frac{\cos(\alpha - \sigma)}{\sin\alpha}; \qquad b = \frac{q\,r_2}{i^2_2}\,\frac{\cos(\alpha + \tau)}{\sin\alpha}$$

$$c = \frac{p\,r_1}{i^2_1}\,\frac{\cos\sigma}{\sin\alpha}; \qquad d = \frac{q\,r_2}{i^2_2}\,\frac{\cos\tau}{\sin\alpha}$$

$$A = (1 + d)\cos\alpha - b; \qquad B = (1 + d) - b\cos\alpha;$$

$$C = (1 + a) - c\cos\alpha; \qquad D = (1 + a)\cos\alpha - c;$$

$$E = \sin\alpha\,[(1 + a)(1 + d) - b\,c]; \qquad F = E/\sin\alpha; \quad \mu = \frac{M_2}{M_1}$$

$$v'_{11}\left[\frac{C + F/\mu}{F\cos\alpha - D} + \frac{A - F\cos\alpha}{\mu F + B}\right] = v_1\left[\frac{F/\mu}{F\cos\alpha - D} - \frac{F\cos\alpha}{\mu F + B}\right] + v_2\left[\frac{\mu F}{\mu F + B} - \frac{F\cos\alpha}{F\cos\alpha - D}\right]$$

$$v'_{22}\left[\frac{\mu F + B}{A - F\cos\alpha} + \frac{F\cos\alpha - D}{C + F/\mu}\right] = v_2\left[\frac{\mu F}{A - F\cos\alpha} + \frac{F\cos\alpha}{C + F/\mu}\right] - v_1\left[\frac{F\cos\alpha}{A - F\cos\alpha} + \frac{F/\mu}{C + F/\mu}\right]$$

$$v_{1q} = v'_{11}\,\frac{A}{E} + v'_{22}\,\frac{B}{E}; \qquad v_{2q} = v'_{11}\,\frac{C}{E} + v'_{22}\,\frac{D}{E} \,.$$

As in the previous examples, the precise direction of the average momentum is not known and must be corrected in each case according to the determined direction of u.

11. Energy Balance

The previous considerations hold only for the impact period itself. The subsequent motion after the impact forces stop are given, on the one hand, by the previously determined velocities and directions of motion directly after the impact and, on the other hand, by the external resistive forces on the path to the final vehicle position. These latter can be treated by means of an energy consideration.

In a collision of two vehicles in straight motion, the initial kinetic energy at the instant of contact is $\Sigma E_0 = M_1 \frac{v^2{}_1}{2} + M_2 \frac{v^2{}_2}{2}$.

During the impact, part of the initial energy is dissipated in destruction (permanent deformation) and part is stored temporarily in elastic deformation. During recoil, this latter amount is reconverted into kinetic energy. According to the above considerations, it is relatively small and hard to take into account; therefore energy balances are generally set up under the assumption of a completely inelastic impact so that calculations may be carried on with the linear and angular velocities determined for the instant of maximum compression. For each of the vehicles involved, the kinetic energy of the vehicles involved just after the impact is

$$E' = E_0 - E_v = M \frac{v'^2}{2} + J \frac{\omega'^2}{2} .$$

This remaining energy is dissipated by the work of the external forces on the appropriate paths, $E' = \int W\,ds$.

For practical application to accidents, the energy balance should generally be developed backward, i.e. starting from the work of the resistive forces. This is only possible if sufficient data concerning the motion of the vehicles are available from track marks, measurements or photographs. In the following some examples of this are given.

12. Energy Balance of the Right-Angle Collision

Figure 10a shows the final positions of the vehicles. In the present case the vehicle velocities were known quite precisely: they were $v_1 = 12$ and $v_2 = 8$ m/sec. The masses of the two vehicles were approximately the same, m = 100 kg sec^2/m. The lever arm is taken as p = 2 and the radius of gyration as i = 1.3 m.

Thus the total initial kinetic energy was

$$E_0 = \frac{m}{2}\,(v^2_1 + v^2_2) = 50\,(144 + 64) = 10\,400 \text{ mkg.}$$

Directly after the impact, vehicle 2 has a translational velocity $v'_2 = u = \sqrt{6{,}17^2 + 6{,}0^2}$ m/s; no rotation results from the collision. Its energy directly after the impact is thus $E'_2 = 50\,(6{,}17^2 + 6{,}0^2) = 3\,704$ mkg.

The translational velocity of the c.g. of vehicle 1 just after the impact is $v'_1 = \sqrt{6{,}0^2 + 1{,}83^2}$ m/s, the angular velocity about the c.g. is $W_1 = 2.17$. Thus, directly after the impact, vehicle 1 still has an energy of

$$E'_1 = 50\,(6{,}0^2 + 1{,}83^2 + 1{,}3^2 \cdot 2{,}17^2) = 2\,365 \text{ mkg.}$$

Since both drivers were severely injured, they could no longer take an active part in the further motion of the vehicles; the directions of motion are so strongly altered in the impact that the steering system is ineffective; thus the c.g. motion can be assumed to be rectilinear. For vehicle 1 the friction coefficient for sideslip on the damp asphalt of the road can be taken as $\mu_g = 0.35$; it is somewhat greater for vehicle 2 $\mu_g = 0.38$ since it moved partly on the rougher sidewalk.

When the left front wheel strikes the curb, it receives a rotation of about 45°. The rotation of vehicle 1 is about 160°. The distances traversed to their final conditions were 9.0 m for vehicle 1 and 10.2 m for vehicle 2. The sideslip and roll components are determined by decomposition into the original direction of motion and normal to it. The roll friction coefficient is taken as $\mu_r = 0.02$. Thus we get for

Vehicle 1

Work of Rotation	1000	0.35	3.5	=	2225
Work of Sideslip	1000	0.35	2.9	=	1015
Work of Rolling	1000	0.02	8.8	=	176
				=	3416 mkg

Vehicle 2

Work of Rotation	1000	0.38	1.9	=	742
Work of Sideslip	1000	0.38	7.1	=	2698
Work of Rolling	1000	0.01	9.2	=	144
Lifting Work of Curb					
Impact Loss at Curb about	1000	0.15		=	150
				=	3734 mkg

13. Oblique Impact of Motor Car and Rail Vehicle

Figure 13 is the vector diagram of the momentum for the collision of a bus with a local train on an unrestricted crossing. The bus was struck approximately in the middle by the locomotive and carried along over a known distance s. After the crossing, the underside of the bus skidded along the rails, and the front and rear wheels hung free on both sides of the raised track bed. The speed of the motor vehicle was known; the problem was to determine whether the train had exceeded the prescribed speed limit at the crossing.

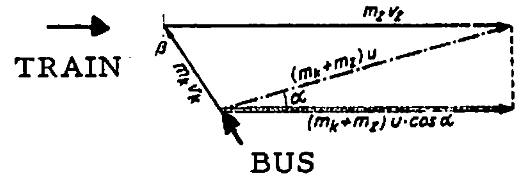

Figure 13 Oblique Collision of Motor Car and Rail Vehicle; Sector Diagram of Momenta

In the vector diagram the common resultant velocity is given by the quantity $(m_k + m_z) u \cdot \cos\alpha$ since the direction of motion is constrained by the rails.

The known distance s is traversed while overcoming resistances which dissipate the kinetic energy remaining after the impact. They consist, for example, of braking or climb resistance, rubbing against the rims, lifting work in derailing axles, drag of trailing machine component sliding along the rails and the sliding resistance of the bottom of the bus against the rails. (In analyzing the accident these were actually considerable differences of opinion relative to the question of to what extent there was a lubricating effect as a result of the machine and diesel oil, water, blood and coal dust spilled during the accident.)

The proof starts by setting up as precise as possible an energy balance which must be adjusted so that the actually determined distance is reproduced. The initial kinetic energy of the train which is required for this yields the desired train velocity.

There are two criticisms of this line of reasoning: first, the common velocity $u \cdot \cos\alpha$ assumes a completely inelastic collision. If, however, the impact is partially elastic, then the much larger train mass attempts to accelerate the smaller bus mass. Second, the assumption of a compact train mass m_z is arguable. The train is more nearly a chain of elastically coupled masses. In the first instant of the impact, only the front mass of the locomotive is effective; after that a considerable part of the kinetic energy is successively stored in the buffer springs mainly in the form of reversible stress energy; accordian type of oscillations are set up in the train.

These transformations of the actual process relative to the assumed inelastic impact result in a weakening of the impact peaks and an increase in impact duration; added to this is the mobility of the masses which are not fixed to the vehicles (water, coal, passengers); at first they fly more or less freely forward and only later, namely when they strike something, do they take part in the impact effect.

In the present case, the indicated calculation gave an irreversibly transformed energy which was obviously too large. For given resistances and free paths, the initial kinetic energy and hence the initial velocity of the train come out too large.

The measured slide path was $s = 50$ m; the mass of the train with the contribution of the rotating masses was $m_z = 15{,}000$ kg sec^2/m; the mass of the bus was $m_k = 1000$ kg sec^2/m; motor vehicle speed was $v_k = 18$ m/sec; roll resistance of the train was 2.5 kg/t. Assume the collision took place with a freely rolling train on a 6% grade at an angle of 125°. The resistance of the machinery parts which hung down is taken as 400 kg. If we assume, for a first cut, that all forces including sliding resistance between the motor vehicle floor and the rails are constant, then the rectangle of the sum of the forces over the slide distance s represents the total dissipated energy E'. The potential energy due to the grade is regarded as drive force with reversed sign. The constant path-average of the deceleration of the joined masses is then $b_m = P/(m_z + m_k)$.

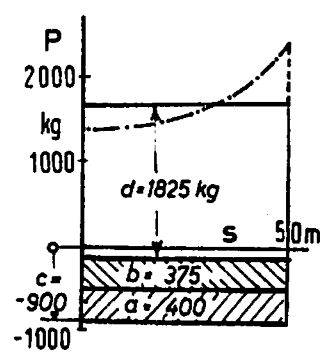

Figure 14 Oblique Collision of Motor Vehicle and Railroad Train, Energy Balance

In the vector impact diagram we have

$$(m_z + m_k)\,u' = m_z v_z - m_k v_k \sin 35° \quad \text{or} \quad 16000 \cdot u' = 15000 \cdot v_z - 1000 \cdot 18 \cdot 0{,}5736$$

$$\text{or} \quad u' = u \cdot \cos a = 0{,}937\, v_z - 0{,}645.$$

Now, if the train had the prescribed value of $v_z = 15$ km/hr = 4.17 m/sec, then the speed of the joined masses directly after the collision would be

$u' = 3.91 - 0.645 = 3.265$ m/sec. To achieve the slide distance $s = v^2/2b = 50$ m, we would need a deceleration of path-mean value $b_m = 0.1064$ on an average sum of resisting forces of $P_m = 16,000 \quad 0.1064 = 1700$ kg. After subtracting the remaining, known applied forces, we thus get for the force of the sliding resistance between the bus floor and the tracks 1825 kg. The question of whether the train exceeded its permissible speed limit is reduced by this consideration to the indication of whether the sliding resistance was greater than the determined value of 1825 kg. Since this resistance obviously depends on velocity, a stepwise integration would be required for a more accurate calculation.

For this we must have an experimentally or theoretically determined curve of friction coefficient μ as a function of sliding velocity v which takes into account the given lubrication conditions. The coefficient of friction decreases with increasing speed, and more rapidly for lubricated than for dry surfaces. The friction force is $P = G\mu$; for the postulated initial velocity of v = 15 km/hr = 4.17 m/sec we get the appropriate friction coefficient from the curve $\mu = f(v)$. If we take this or a slightly higher value for a short time interval as a constant, then all resistances are known; the velocity decrease caused by them in this time interval can be calculated.

For the next small time interval a new value of μ is taken from the curve of $\mu = f(v)$ for the lower velocity, and the calculation is carried along until the body comes to rest. If the total skid distance determined in this way in the example is not less than the measured distance (50 m) or the path-mean value of all resistances is not greater than 1825 kg, then the initial velocity of the train was not greater than 15 km/hr, which was to be proved Figure 14.

SUMMARY

The theoretical fundamentals of the impact process were first developed. Then - in part by means of practical examples - it was shown that a combined application of impact laws (vector diagram of momenta) and of the law of conservation of energy (Energy balance of resistance energy after impact) makes possible in many cases a complete explanation of the motions and velocities in the collision of motor vehicles. The reliability of the assumptions made in this process is thereby reinforced.

REFERENCES

1. Eberan von Eberhorst: Dringliche Sicherheitsprobleme der Kraftverkehrstechnik (Pressing Safety Problems of Traffic Engineering), ATZ 1958, pages 299-308.

2. Brüderlin, Ad.: Die Mechanik des Verkehrsunfalls bei Kraftfahrzeugen (The Mechanics of Traffic Accidents for Motor Vehicles), Zürich 1941.

3. Lossagk, H.: Der Zusammenstoss (The Collision), BTU 1958, pages 256-260.

4. Bergen, F.: Das Gesetz des Kraftverlaufs beim Stoss (The Law of Force Behavior in an Impact), Braunschweig 1924.